Flows on the Space of Poisson Strutcures

Research Report

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Flow Construction

In this project, the main concern is studying and computing certain flows on the space of Poisson structures and checking their (non)triviality. To understand this, the following background information is needed.

A *Poisson Bracket* on a manifold M is a bilinear operation:

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \tag{1}$$

such that for all $f, g, h \in \mathcal{C}^\infty(M)$ the following properties hold:

1. *Skew-symmetry*: $\{f, g\} = -\{g, f\}$
2. *Derivation rule*: $\{f, gh\} = \{f, g\}h + g\{f, h\}$
3. *Jacobi identity*: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$

At any point p in M, $\{f, g\}$ depends on the first derivatives of f and g at p (property 2). We can represent a Poisson bracket by a Poisson bivector $\mathcal{P}$ which has the following form:

$$\mathcal{P} = \sum_{i,j} \mathcal{P}^{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{2}$$

where $\mathcal{P}^{i,j} = \{x_i, x_j\}$ are smooth functions.

A *Kontsevich quantization graph* is a directed graph Γ with the set of vertices $V_\Gamma = \{1, \cdots, n\} \sqcup \{L, R\}$ and $2n$ edges, where $\{1, \cdots, n\}$ is the set of internal vertices and $\{L, R\}$ is the set of external vertices. The edges are labelled, the out degree of each internal vertex is 2 and the out degree of each external vertex is 0. See graph in Figure 1.

Kontsevich [2] associates a *bidifferential operator* $B_{G,\alpha}$ to a graph G and a bivector α. The bidifferential operator sums over all the possible labellings of the edges of G where each term is a product of partial derivatives that correspond to the edges.

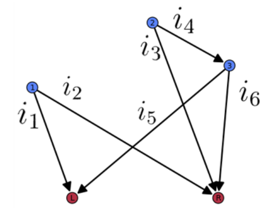

Figure 1: A Kontsevich quantization graph (n=3).

Example 1: Consider the graph in Figure 1. The bidifferential operator associated with it is

$$(f, g) \mapsto \sum_{i_1,\cdots,i_6} \alpha^{i_1 i_2} \alpha^{i_3 i_4} \partial_{i_4}(\alpha^{i_5 i_6}) \partial_{i_1} \partial_{i_5}(f) \partial_{i_2} \partial_{i_3} \partial_{i_6(g)}$$

Take an unoriented graph G, and orient G in all possible ways that would give a Kontsevich quantization graph after adding the external vertices L and R. Summing over the orientations $\Gamma = \sum_i w_i \Gamma_i$ will give an ordinary differential equation (flow) on the space of bivectors:

$$\frac{d\mathcal{P}_t}{dt} = \mathcal{F}_G(\mathcal{P}_t) = \sum_i w_{\Gamma_i} B_{\Gamma_i}(\mathcal{P}) \tag{3}$$

It was observed by Kontsevich that for certain special linear combinations of unoriented graphs G called *graph cocycles*, this flow preserves the space of Poisson structures i.e. if $\mathcal{P}_t$ is a solution and $[\mathcal{P}_0, \mathcal{P}_0] = 0$, then $[\mathcal{P}_t, \mathcal{P}_t] = 0$ for all t.

Example 2: The simplest graph cocycle is given by taking G to be the wheel graph with 3 spokes which will produce the graphs in Figure 2. Given a Poisson bivector $\mathcal{P}$, the flow equation (3) becomes:

$$\frac{d\mathcal{P}_t}{dt} = \mathcal{F}_G(\mathcal{P}_t) = w_{\Gamma_1} B_{\Gamma_1}(\mathcal{P}) + w_{\Gamma_2} B_{\Gamma_2}(\mathcal{P}) = 24 B_{\Gamma_1}(\mathcal{P}) + 8 B_{\Gamma_2}(\mathcal{P})$$

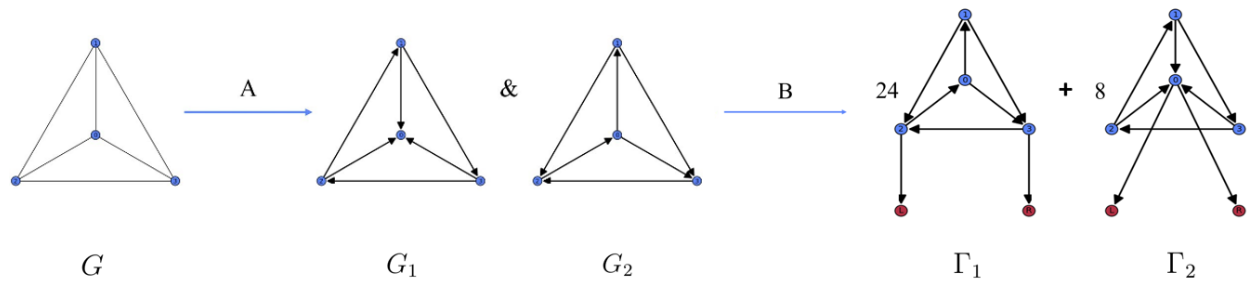

Figure 2: ***A***: orienting G in all possible ways which gives G_1 and G_2. ***B***: adding the vertices L and R to the graphs G_1 and G_2 obtained by ***A*** such that the resulting graphs are Kontsevich quantization graphs of the tetrahedral graph cocycle; Γ_1 and Γ_2 respectively.

We say that a solution to the flow equation (3) is *trivial* if $\mathcal{P}_t$ is obtained from its initial state by a t-dependent change of coordinates. The triviality is equivalent to saying that $\mathcal{F}_G(\mathcal{P}_t)$ is a Poisson coboundary which means that there exist a vector field V_t such that

$$\mathcal{F}_G(\mathcal{P}_t) = [\mathcal{P}_t, V_t] \tag{4}$$

where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket.

The research problem is to determine the (non)triviality of the flows.

Results

Calculating the flow using the formula given in equation (3) is nontrivial even for the simple case of the tetraherdral graph cocycle. Therefore, to compute the flow, we used a previously written SageMath package [1] to represent the Poisson bivectors and the Kontsevich quantization graphs and to compute their bidifferential operators. We wrote the code with the package mentioned because we tested the SageMath manifolds package and it turned out to be much slower. The flow bivector is essentially represented as a matrix. One of the difficulties was to generate the graphs and compute their weights in a relatively fast way. Therefore, we used a program called nauty [4] to orient the graphs along with methods from the graphs package of SageMath. To avoid long run-time, and since such graph cocycles are known, we saved their respective oriented Kontsevich quantization graphs and weights in a database. Also, to save time when trying to write the commands to calculate the flow, the code uses another database that includes the graph cocycles.

We applied the code to calculate the flow of some examples and most of the resulting flows were identically zero but in one interesting case, the computed flow was non zero even though it is known that it is a coboundary [3]. Now, we are developing a method that checks whether or not a flow is a coboundary. After that, it will be very interesting to test more examples.

References

[1] Peter Banks.

[2] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[3] Cesar Lecoutre and Susan J. Sierra. A new family of Poisson algebras and their deformations. *Nagoya Math. J.*, 233:32–86, 2019.

[4] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, {II}. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014.